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A NUMERICAL STUDY OF BIFURCATION IN LAMINAR FLOW IN CURVED DUCTS

S. JAYANTI AND G. F. HEWITT

Department of Chemical Engineering and Chemical Technology, Imperial College of Science, Technology and Medicine, London SW7 2BY, U.K.

SUMMARY

The bifurcation phenomenon whereby multiple-vortex secondary flow occurs in place of the normal two-vortex flow in laminar flow in curved ducts has previously been studied numerically by several researchers. However, the various results have been conflicting on many points. The present paper describes a set of numerical experiments conducted to study the effect of numerical accuracy on the solution. The results show that the transition from two- to four-vortex structure depends strongly on the differencing scheme and to a lesser extent on the grid size. The study also shows that as the Reynolds number of the flow increases, a two-vortex structure is re-established via a path which involves strongly asymmetric secondary flow patterns. These results are in agreement, at least qualitatively, with recent experimental theoretical and numerical results.

KEY WORDS Bifurcation Laminar flow Curved ducts Numerical accuracy Finite difference methods Secondary flows Asymmetry

1. INTRODUCTION

The secondary flow in curved ducts normally consists of two counter-rotating symmetric vortices. However, above a certain critical Dean number (of about 100), one or more pairs of additional vortices may be formed near the outer wall of the curved duct. This phenomenon was first observed in the numerical calculations of Akiyama[1] and the first experimental proof was provided by Joseph *et al.*[2] Since then, there have been a number of numerical studies of the flow field.[2–12] In spite of these and other studies, the qualitative and quantitative nature of the phenomenon are yet to be understood fully, prompting some to classify this among the unanswered questions in fluid mechanics.[13] The main reason for this is the lack of consensus among the various numerical results as explained below.

The bifurcation phenomenon is sometimes attributed[5,6] to centrifugal instability. Stuart[14] discusses several types of centrifugal instability; one of these is that which would occur in a two-dimensional curved channel between which fluid flows under the action of a pressure gradient. Dean[15] studied this case theoretically and concluded that the instability would occur at a Dean number ($Re(d/D)^{1/2}$) of 25·5. This perhaps is too small for it to be the 'cause' of the bifurcation, which has so far been reported only for Dean numbers greater than about 100. The more serious concern is that the results of numerical studies are not consistent on many points. The critical Dean number at which a four-vortex secondary flow first appears has been reported to be about 105 by Joseph *et al.*[2], 202 by Cheng *et al.*[5], 143 by Ghia and Sokhey[6] and 125 by Ghia *et al.*[11] However, experimental results[2,16] put this transition to be at a Dean number of about 100, which

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is consistent with the more recent numerical calculations of Masliyah,[7] Dennis and Ng[8] and Nandakumar and Masliyah[9] in ducts of various cross-sections. However, Hille *et al.*[17] found experimentally that the four-vortex structure was present only for Dean numbers between 150 and 300. This is consistent qualitatively with the numerical results of Cheng *et al.*[5] who obtained the four-cell structure for Dean numbers between 202 and 520, but is in conflict with the results of Ghia *et al.*[11] who carried out their calculations up to a Dean number of 900 and found that the four-cell structure persisted.

Another issue under debate is the stability of the multiple-vortex solution. Winters[12] obtained, by solving an extended set of Navier–Stokes equations, not only two- and four-cell structures but also asymmetric solutions at the same axial pressure gradient in fully developed flow in a coiled tube of square cross-section. He analysed the stability of these solutions and concluded that all multiple solutions except the two-cell structure are unstable. A similar conclusion has been reported by Goering *et al.*,[13] who used perturbation analysis. (Yang and Keller[10] also obtained multiple-vortex solutions (including an eight-vortex system) in a tube of circular cross-section, but they imposed a symmetry boundary condition along the horizontal axis of the tube and, as will be seen later, this may limit the applicability of their results.) Of all the previous studies, only the experimental measurements of Hille *et al.*[17] show an asymmetric solution, while Winters[12] was the only one to obtain a weakly asymmetric solution numerically.

Thus several questions about the nature of the flow remain unanswered (as pointed out earlier by Goering *et al.*[13]): (i) Is there a critical Dean number at which the transition from two- to four-cell structure occurs? (ii) What happens at high Dean numbers: does the four-cell structure persist or is a two-cell structure re-established? (iii) If a two-cell structure is re-established, how does the transition from four- to two-cell structure take place? Does it involve asymmetric solutions? To these questions on the nature of the flow can be added another on the nature of solution method. (iv) Why is there so much discrepancy among the various numerical results? How sensitive is the solution to the accuracy of the numerical scheme?

The purpose of the work described in the present paper was to attempt to answer these questions by conducting carefully controlled '*numerical*' experiments designed specifically to address these questions. These are described in the next section. The results of these experiments are given in Section 3 and the main conclusions from the study are discussed in Section 4.

2. SET-UP OF NUMERICAL EXPERIMENTS

In these experiments we wish to investigate (a) the existence of a critical Dean number for the appearance of a four-vortex secondary flow structure in curved tubes, (b) the nature of the flow at high Dean numbers, including the possibility of asymmetric solutions, and (c) the sensitivity of the solution to numerical accuracy, specifically to the grid size and (d) to the discretization scheme.

2.1. Critical Dean number

The existence of a critical Dean number is investigated by calculating the critical Reynolds number at which the transition occurs in curved ducts of three different diameter ratios: (i) a coiled tube of coil-to-duct diameter ratio of 50, (ii) a U-bend of bend-to-duct diameter ratio of 10 and (iii) a U-bend of diameter ratio of 20. For ease in generating the grid for numerical calculations using the HARWELL-FLOW3D computer program (see Section 2.5), the duct is taken to be of square cross-section. It is noted in passing that the coil geometry corresponds to one of the cases investigated numerically (using finite element methods) by Winters[12] and that the flow through the two bends has been investigated experimentally by Ohba and co-workers.[18,19]

2.2. Nature of flow at high Dean numbers

This is investigated by carrying out the calculations up to a high Reynolds number in each duct. In view of the possibility of asymmetric solutions, the flow through the full duct is simulated. The stability of any asymmetric solutions is studied by starting from various initial solutions. The effect on an asymmetric solution of imposing a symmetry plane at the midplane of the duct (thereby simulating the flow through only half the duct) is also studied.

2.3. Grid dependence of the solution

The grid dependence of the solution is investigated by calculating the flow structure in the coiled tube on three grids: a 20×20 grid, a 30×30 grid and a 40×40 grid. In view of the excessive computational time required for three-dimensional flows, only two grids, namely a $32 \times 20 \times 20$ grid and a $38 \times 30 \times 30$ grid, for the U-bend of diameter ratio of 20 and only a $32 \times 20 \times 20$ grid for the other U-bend are used in the calculations. A numerical grid generation package associated with the FLOW3D code was used to generate all the grids.

2.4. Sensitivity of solution to differencing scheme

The effect of the discretization scheme is studied by calculating the flow field using three differencing schemes: the hybrid differencing scheme, the higher-order upwinding (HUW) differencing scheme and the QUICK differencing scheme based on quadratic upstream interpolation. These schemes are briefly described below with reference to discretization of the advection term $u\partial\phi/\partial x$.

The hybrid differencing scheme is based on a combination of the second-order-accurate central differencing scheme and the first-order-accurate upwind differencing scheme. For a mesh Reynolds number, defined as $u\Delta x/\nu$, between -2 and 2 the central differencing scheme is used:

$$u\frac{\partial\phi}{\partial x}=\frac{u}{2\Delta x}(\phi_{i+1}-\phi_{i-1}). \tag{1a}$$

Otherwise, an upwind differencing scheme is used:

$$u\frac{\partial\phi}{\partial x}=\frac{u}{\Delta x}(\phi_{i}-\phi_{i-1}), \quad u>0, \tag{1b}$$

$$u\frac{\partial\phi}{\partial x}=\frac{u}{\Delta x}(\phi_{i+1}-\phi_{i}), \quad u<0. \tag{1c}$$

The use of the less accurate upwind differencing is necessary when the absolute value of the mesh Reynolds number is greater than 2, because the scheme then loses its diagonal dominance and is subject to numerical instabilities.

The higher-order upwind differencing scheme[20] is second-order-accurate and does not lead to 'wiggles' (spatial oscillations), and can be described as follows:

$$u\frac{\partial\phi}{\partial x}=\frac{u}{2\Delta x}(3\phi_{i}-4\phi_{i-1}+\phi_{i-2}), \qquad u>0, \tag{2a}$$

$$u\frac{\partial\phi}{\partial x}=\frac{u}{2\Delta x}(-3\phi_{i}+4\phi_{i+1}-\phi_{i+2}), \qquad u<0. \tag{2b}$$

The QUICK scheme[21] is third-order-accurate and exhibits only bounded wiggles. It is described as follows:

$$u\frac{\partial\phi}{\partial x}=\frac{u}{6\Delta x}(2\phi_{i+1}+3\phi_i-6\phi_{i-1}+\phi_{i-2}), \qquad u>0, \tag{3a}$$

$$u\frac{\partial\phi}{\partial x}=\frac{u}{6\Delta x}(-2\phi_{i-1}-3\phi_i+6\phi_{i+1}-\phi_{i+2}), \qquad u<0, \tag{3b}$$

2.5. Numerical solution procedure

The HARWELL-FLOW3D computer program[22] is used for numerical solution of the governing partial differential equations. It uses a finite difference (volume) method on a general non-orthogonal body-fitted grid and has a polyalgorithmic structure whereby options are available for the user to select from different discretization schemes and solution algorithms. The Rhie–Chow algorithm[23] extended to three dimensions[24] is used to overcome the problem of chequerboard oscillations usually associated with the use of non-staggered grids. In the present calculations the SIMPLEC algorithm[25] is used for pressure and velocity decoupling. More details of the computer programme can be found in References 22, 24, 26 and 27.

3. RESULTS AND DISCUSSION

The flow field was calculated over a wide range of Reynolds number for various combinations of duct geometry, grid size and discretization scheme. In view of the sheer volume of output from these calculations, our attention is restricted to studying the effect of various parameters on the secondary flow vectors at the outlet (fully developed flow) in the case of the coiled tube and at the bend exit in the case of the two bends. Also, the emphasis is on the qualitative nature of the flow field and any quantitative aspects are neglected.

First we examine typical results from a series of calculations in which the Reynolds number of the flow increases but other parameters (e.g. grid, numerical method, etc.) remain the same. Figure 1 shows the secondary flow pattern at the exit of the bend (with a diameter ratio of 20) at Reynolds numbers of 200, 400, 580, 1200, 1400, 1600, 1800 and 2000. The grid size in these calculations was $32\times20\times20$ and the higher-order upwind differencing scheme was used. It is seen that at a Reynolds number of 200 the secondary flow consists of a pair of symmetric vortices. This structure is retained at $Re=400$, though the flow pattern near the outer wall begins to change. At $Re=580$ an additional pair of symmetric vortices is created near the outer wall. The four-vortex symmetric structure is maintained up to $Re\approx1200$, where it begins to become asymmetric. It is clearly asymmetric at $Re=1400$; one of the two smaller vortices appears to be moving towards the corner and the other appears to get bigger. This process is completed by $Re\approx1600$, where only three vortices are evident. As Re increases further to 1800, this vortex is also entrained by one of the two main vortices and a two-vortex pattern is established at $Re\approx2000$.

This is the basic flow pattern as the Reynolds number increases. In the rest of the section the main interest is on how it changes as other parameters of the solution (and not necessarily of the flow field itself!) such as the curvature ratio and the grid size are changed.

3.1. Transition from two- to four-vortex structure

The Reynolds number or Dean number at which the two-vortex structure changes to a four-vortex one can be determined by increasing Re in small steps around the point of transition. In

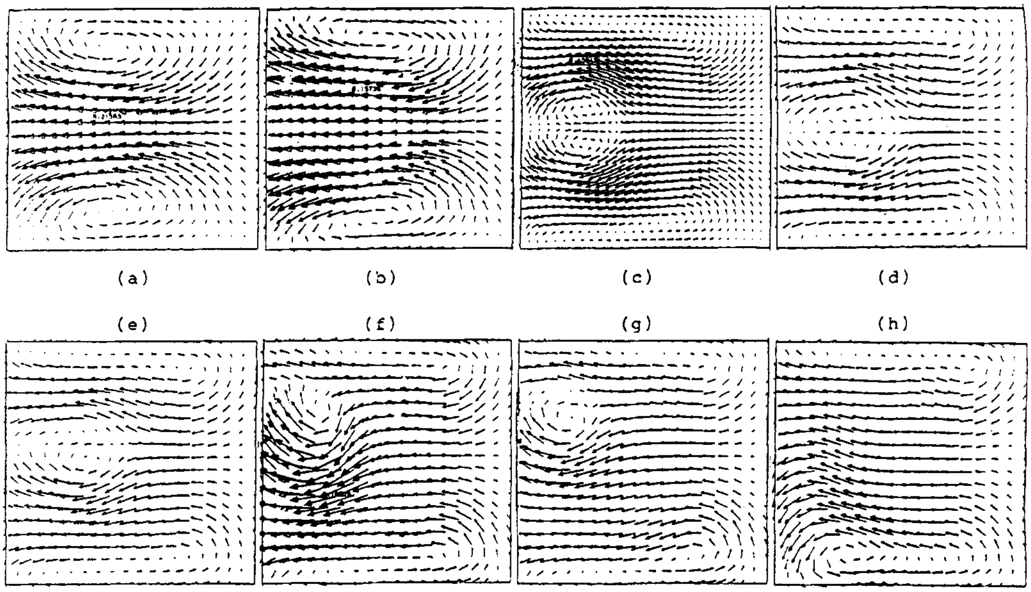


Figure 1. Secondary flow pattern at exit from U-bend with diameter ratio of 20 at Reynolds numbers of (a) 200, (b) 400, (c) 580, (d) 1200, (e) 1400, (f) 1600, (g) 1800 and (h) 2000. The grid size is 32 × 20 × 20 and the differencing scheme the higher upwind differencing (HUW)

the case of the long-radius bend this was done by calculating the flow field at Reynolds numbers of 400, 450, 475 and 500 using HUW as the differencing scheme. The flow pattern at the exit showed a two-vortex structure for $Re = 400$, 450 and 475 but a four-cell structure at $Re = 500$; thus the transition Reynolds number is between 475 and 500 for this set of conditions. Figure 2 shows the secondary flow pattern calculated at $Re = 475$ using HUW and QUICK on a *finer* grid. A clear four-cell structure is evident in the flow field calculated using QUICK, which is more accurate than HUW. Also, increasing the grid size from 20×20 to 30×30 in the duct cross-section does not seem to have a significant effect on the critical Re in this case. Thus the transition can be supposed to occur at $Re = 475$.

The critical transition for the short-radius bend (diameter ratio of 10) can be obtained from Figure 3, which shows the secondary flow at Reynolds numbers of 350, 450 and 580 calculated on a 20×20 grid using the QUICK differencing scheme. It is seen that at $Re = 350$ the solution using QUICK is on the verge of transition to a four-vortex structure, and the transition Reynolds number can be set to 350 for this bend. The sensitivity of the solution to the differencing scheme can be gauged from the flow structures shown in Figure 4, which are calculated at the same Re but using the less accurate HUW and hybrid differencing schemes. The HUW solution shows a four-cell structure at $Re = 450$ but not at $Re = 350$, while the hybrid solution retains a two-vortex structure even at $Re = 580$. A calculation at $Re = 1000$ shows it to have a four-cell structure. Thus we find that the transition Reynolds number for a given curved duct depends very much on the differencing scheme used. This is especially true of the hybrid differencing scheme; HUW and QUICK seem to give fairly consistent results.

The calculation of the fully developed flow in the coil should take less than the developing flow calculations in the bends, which need a three-dimensional grid. It was therefore hoped that this would allow a systematic study of the effect of the grid size and the differencing scheme in a wider range. However, the solution was found to be extremely difficult to converge near the transition

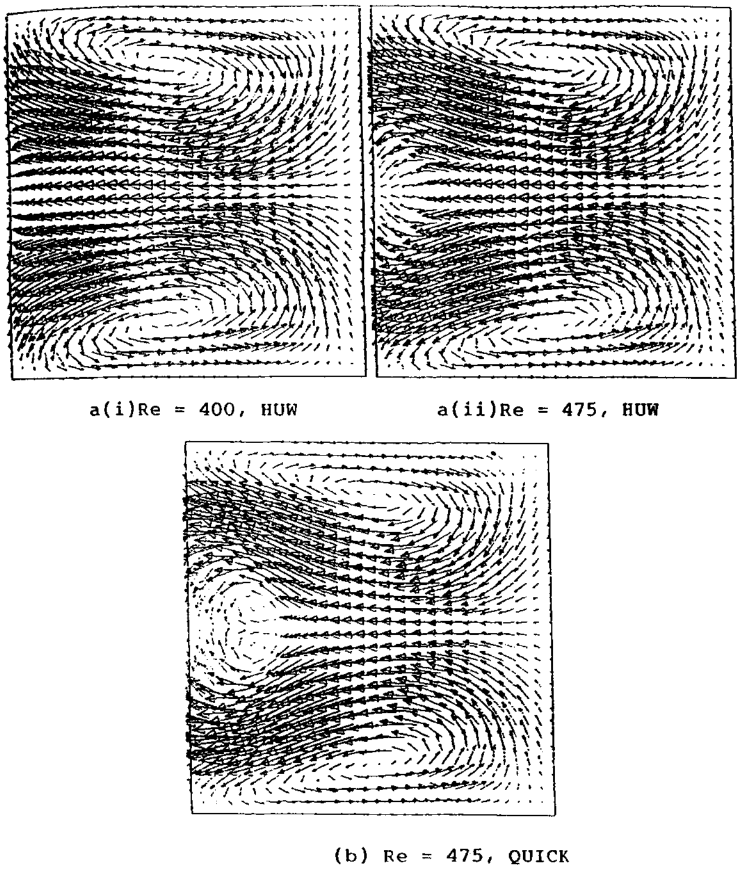


Figure 2. Secondary flow pattern at $Re=475$ at bend exit calculated on a $38\times30\times30$ grid using (a) HUW and (b) QUICK differencing schemes

point as explained below; this limited the range of parameters investigated, but the results were no less interesting.

The coarse grid (20×20) solution using all three differencing schemes (hybrid, HUW and QUICK) had a two-vortex symmetric structure at a Reynolds number of 600. Both the hybrid and HUW solutions were of the symmetric, two-vortex type at $Re=700$. However, the QUICK solution did not converge easily and was found to be oscillating. Although it appeared to have a four-vortex structure at the end of 800 iterations, it gradually tended towards a two-vortex solution at the end of 2000 iterations. At this point the solution was still oscillating, but only with a relatively small amplitude. The solution at various points during an oscillation showed only slightly asymmetric two-cell structure; thus it was concluded that the solution would finally converge to a two-vortex solution. However, when the Reynolds number was increased to 800, the solution oscillated between four-vortex solutions only. Thus the transition between two- and four-vortex solutions occurs at a Reynolds number of about 800 for this coil.

In order to determine the effect of numerical accuracy, the calculations were repeated on finer grids and also using the HUW and hybrid schemes. At $Re=800$ the HUW solution on a coarse grid had a (non-converged) four-cell structure after 800 iterations but converged to a two-cell

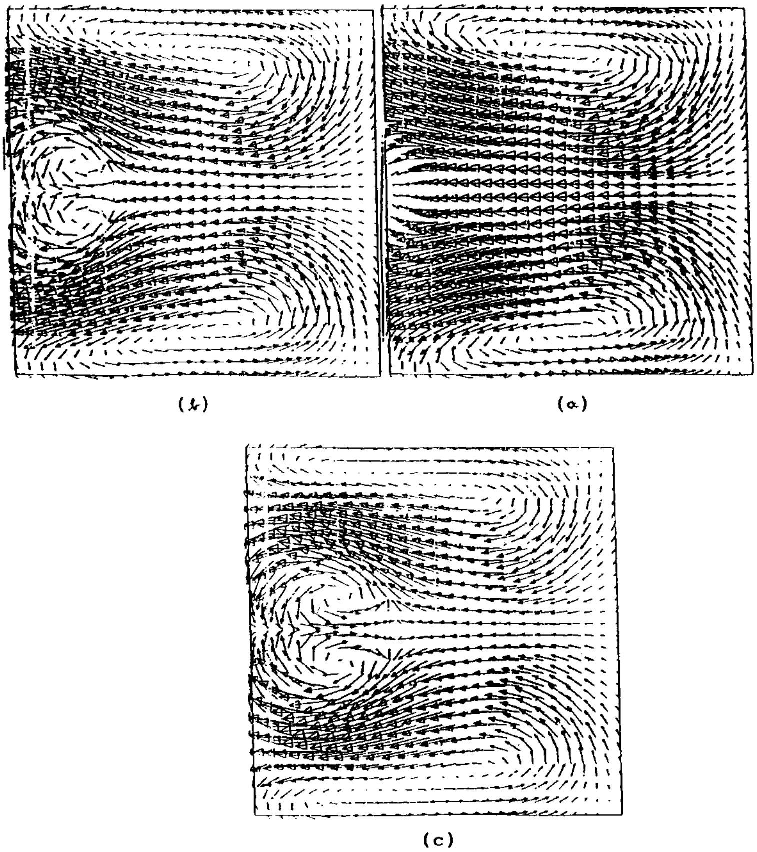


Figure 3. Secondary flow patterns at exit of short-radius bend (diameter ratio of 10) calculated using QUICK scheme at Reynolds numbers of (a) 350, (b) 450 and (c) 580

structure at the end of some 2000 iterations. The solutions did not change appreciably when the grid size was increased to 30×30. However, at a grid size of 40×40 the solution using HUW appeared to oscillate roughly between two extrema. The solutions at the two extrema and at the approximate midpoint of the oscillation are shown in Figure 5. It can be seen that the solutions oscillate between a four- and a two-cell structure. These oscillations were not damped even after 2000 iterations.

The set of calculations using the less accurate hybrid differencing scheme also followed the same trend. On the coarse grid the solution converged to a two-cell structure in 800 iterations. However, on the finer 40×40 grid it initially acquired a four-vortex structure which decayed to a two-vortex one after 3000 iterations. The solution at various stages in this process is shown in Figure 6.

We clearly see the effect of numerical accuracy on the solution. The solution using the QUICK scheme, which is third-order-accurate, starts oscillating at a Reynolds number of 700 and acquires a four-cell structure at $Re = 800$ on a coarse grid. At this stage the HUW solution, which is second-order-accurate, shows signs of oscillation but converges to a two-cell pattern. When the grid is refined, the HUW solution now starts to oscillate between a four- and a two-cell structure. For the same Reynolds number the hybrid scheme solution gives a two-vortex structure within

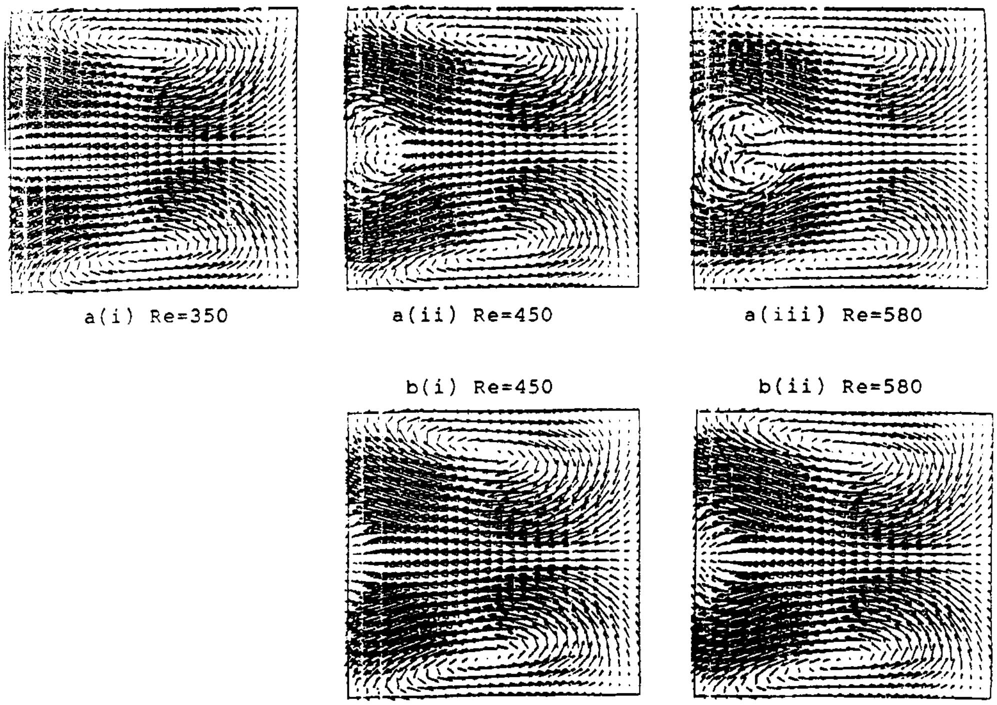


Figure 4. Same as Figure 3 but using (a) HUW and (b) hybrid schemes

800 iterations on a coarse 20 × 20 grid but shows some tendency towards a four-cell structure on a finer 40 × 40 grid, and would probably give a four-vortex structure if the grid size was further increased. Thus the overall numerical accuracy is important; an effectively first-order-accurate scheme such as the hybrid scheme would require a very fine grid to calculate the transition point accurately.

It still remains to explain why no *converged* four-cell structure was ever obtained in the calculations of the flow in this coil. There are two plausible reasons for this. The first is that the solutions would converge if they were carried further than 2500 iterations. A case in point is the hybrid solution on a 40 × 40 grid, which eventually converged (to a two-vortex structure) after about 3000 iterations (see Figure 6). The second and perhaps the more probable explanation is that the flow has multiple solutions after the bifurcation. Winters[12] recently studied the bifurcation characteristics of the same flow (i.e. fully developed flow in a square coil of diameter ratio of 50) by solving an extended system of equations in a finite element approximation. He found that several solutions, including two- and four-cell and symmetric and (weakly) asymmetric solutions, were possible at a Dean number of around 100. The various solutions between which the solution oscillates could represent the solutions along different branches of the bifurcation tree. Lack of computing resources and time did not permit further investigation of these solutions; however, the point is made that the solution (obtained using finite difference methods) can be grid- and differencing-scheme-dependent.

The above calculations show that the critical Reynolds number for this coil is about 800. Since the diameter ratio is 50, the critical Dean number is 113. For the long-radius bend case the critical *Re* was 475 and the curvature ratio 20, giving a critical Dean number of 106. For the short-radius

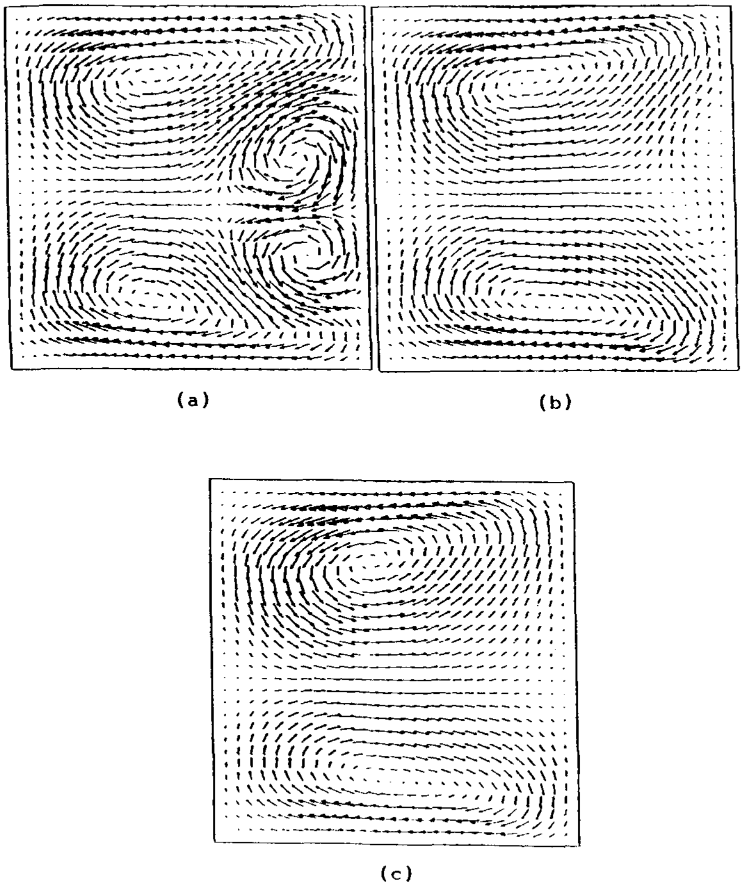
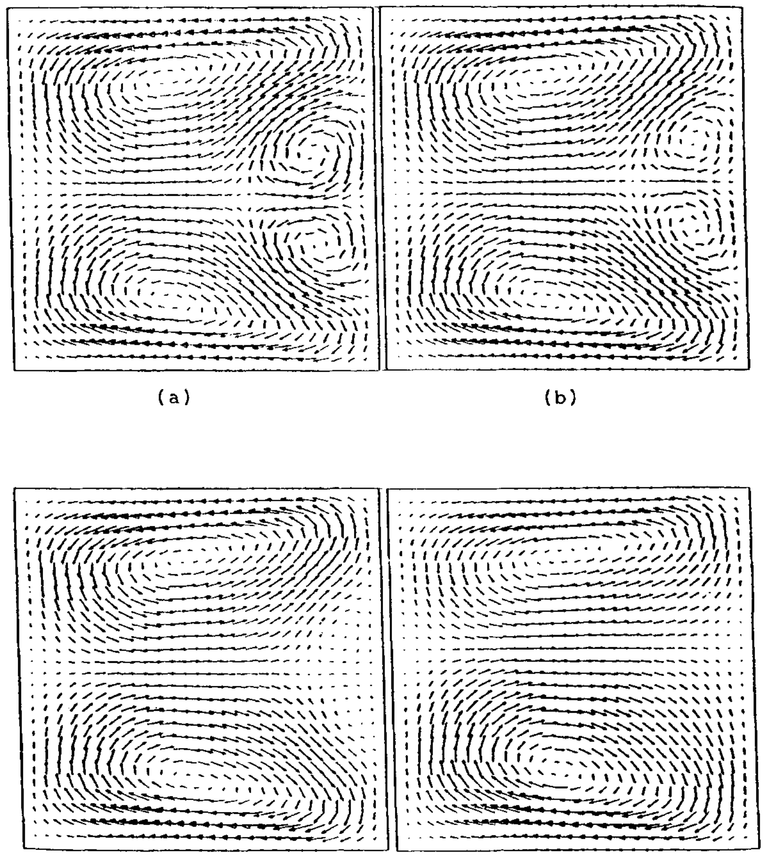


Figure 5. Fully developed secondary flow calculated in coiled tube of square cross-section and diameter ratio of 50 on a 40×40 grid using HUW scheme: (a), (b) at the two extremes of the oscillation; (c) at roughly the midpoint of the oscillation. The Reynolds number of the flow is 800

bend of diameter ratio of 10 the critical *Re* was 350, which corresponds to a Dean number of 110. Thus for all three cases investigated here the critical Dean number at which a four-cell structure appears in curved ducts of square cross-section is about 110. This is consistent with the experimental results of Joseph *et al.*[2] and Cheng and Yuen[16] and with the numerical results of Masliyah[7] and Dennis and Ng,[8] but is in conflict with the results of Cheng *et al.*[5] and Ghia and Sokhey[6] and in rough agreement with the result of Ghia *et al.*[11]

3.2. Appearance of an asymmetric flow pattern

As shown in Figure 1, the symmetric four-vortex pattern may degenerate into an asymmetric pattern as the Reynolds number is further increased. These calculations were done using the HUW differencing scheme. When the hybrid differencing scheme was used, no asymmetric solutions were found even at a Reynolds number of 2100. The secondary flow field consisted of two pairs of symmetric vortices. However, the transition to an asymmetric flow pattern occurred when the more accurate QUICK differencing scheme was used. Here the transition occurred at a lower Reynolds number, which may indicate its sensitivity to numerical accuracy. The flow (see



(a) (b) (c) (d)

Figure 6. Secondary flow in coil calculated using hybrid scheme after (a) 800, (b) 1500, (c) 2200 and (d) 2900 iterations

Figure 7) is symmetric at $Re = 1000$ but becomes asymmetric at $Re = 1100$. It shows a nearly three-vortex structure at $Re = 1150$ and subsequently reverts to a two-vortex-type structure. Converged solutions could not obtained for Reynolds numbers higher than 1300 in this case.

We thus see a strong dependence of the transition to asymmetric flow pattern on the method used. With hybrid differencing no transition was obtained; with HUW the transition occurred at a Reynolds number of about 1300, whereas it was obtained at $Re = 1100$ when the more accurate QUICK differencing was used. The sensitivity of these solutions to the initial guess solution was investigated systematically by using a four-vortex, a two-vortex and an asymmetric solution as the initial guess for many cases. In all cases the final converged solution, if one was obtained at all, was found to be insensitive to the initial guess.

A similar pattern was found in the calculations for the short-radius U-bend (diameter ratio of 10), though only a few cases have been investigated. The results obtained using the QUICK scheme are shown in Figure 8. Here the solution consists of a *six*-vortex symmetric pattern at a Reynolds number 1000! This is brought about by the breaking up of the pair of smaller vortices near the outer wall. As the Reynolds number is increased, the symmetry of the flow is broken; at $Re = 1300$ we see a five-vortex asymmetric structure and at $Re = 1500$ a four-vortex one. Note that

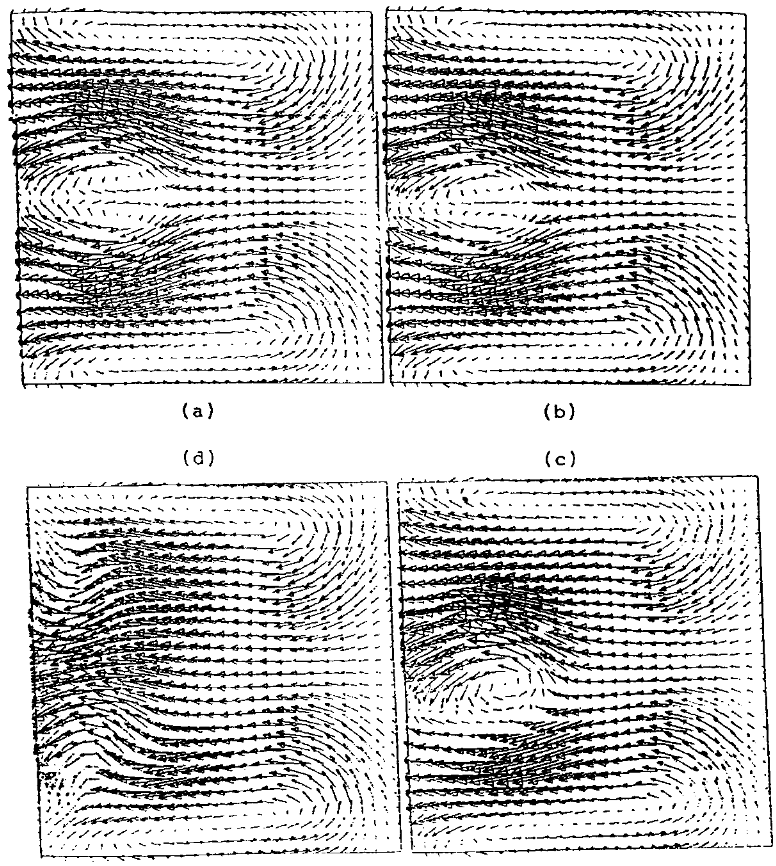


Figure 7. Secondary flow in long-radius U-bend calculated using QUICK scheme at Reynolds numbers of (a) 1000, (b) 1100, (c) 1150 and (d) 1200

symmetric multiple-vortex solutions, including one consisting of *eight* vortices, have been obtained in a circular tube by Yang and Keller;[10] however, these were obtained by imposing a symmetry plane along the horizontal axis.

Thus it appears that an asymmetric secondary flow pattern and multiple-vortex solutions are possible in the laminar flow through curved ducts, as suggested by Winters[12] and Goering *et al.*[13] Winters actually obtained a weakly asymmetric flow pattern in his numerical calculations of the developed flow in a coil. As far as experimental results are concerned, Hille *et al.*[17] obtained an asymmetric secondary flow pattern. One of the additional pair of vortices was smaller than the other, and both appeared to be heading towards one of the corners. This is consistent with the scenario presented in Figure 1, which recurs in various forms as the parameters are changed. At this stage it is perhaps worth mentioning that Ohba *et al.*[19] found velocity fluctuations in laminar flow through a U-bend of square cross-section. These fluctuations (see Figure 9) were present at $Re=670$ and 1340 but not at $Re=167$ or 1500. Inasmuch as the present calculations for this bend geometry show a transition to an asymmetric flow structure at $Re\approx 1000$ (see Figure 7), these fluctuations may be associated with the bifurcation phenomenon. However, the evidence is not conclusive, since our calculations were time-independent and do not indicate the temporal stability of the solution.

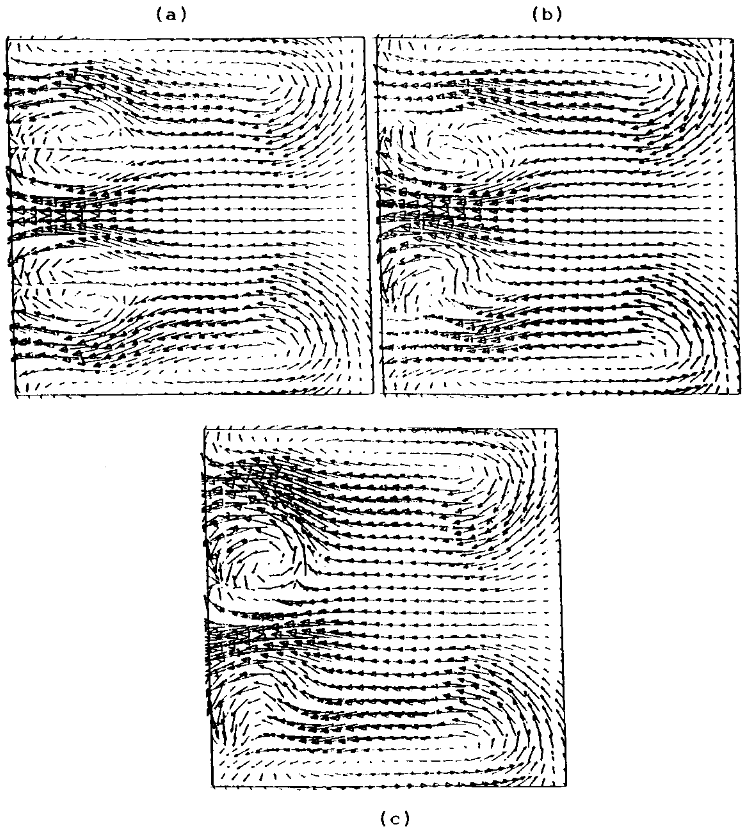


Figure 8. Secondary flow in short-radius U-bend calculated using QUICK scheme at Reynolds numbers of (a) 1000, (b) 1300 and (c) 1500

There are two reasons why the asymmetric patterns have not been reported in earlier studies of this flow. The first is the numerical accuracy, as demonstrated above: the hybrid solution does not show an asymmetric structure whereas the QUICK and HUW solutions do. Another possibility is the imposition of a symmetry boundary condition at the midplane, as was done for example by Joseph *et al.*[2] Cheng *et al.*[5] and Nandakumar and Masliyah.[9] In order to investigate this effect, the calculations in the long-radius U-bend (diameter ratio of 20) were repeated with a symmetry boundary condition at the midplane. In this case only the flow through half the duct was calculated. The results show that the transition Reynolds number from two- to four-vortex symmetric flow is not affected by the imposition of a symmetry boundary condition. However, the half-duct simulation gave a symmetric four-vortex solution even if the solution from the full duct simulation showed a strongly asymmetric secondary flow, e.g. as in Figure 1 at $Re = 1600$. Obviously, the imposition of symmetry in such cases is incorrect. However, the full duct simulation in itself does not necessarily give asymmetric solutions. Ghia *et al.*[11] solved for the fully developed flow in a square coil of diameter ratio of 100 without the symmetry boundary condition, with grid refinement and a second-order-accurate differencing scheme. They found that the four-vortex symmetric solution persisted up to a Dean number of at least 900; this should be compared with the results of Winters,[12] who obtained multiple (including asymmetric) solutions for Dean numbers as low as 80–140 in a coil of diameter ratio of 50, and with those of Cheng *et al.*,[5] who found that a two-vortex structure was re-established after a Dean number

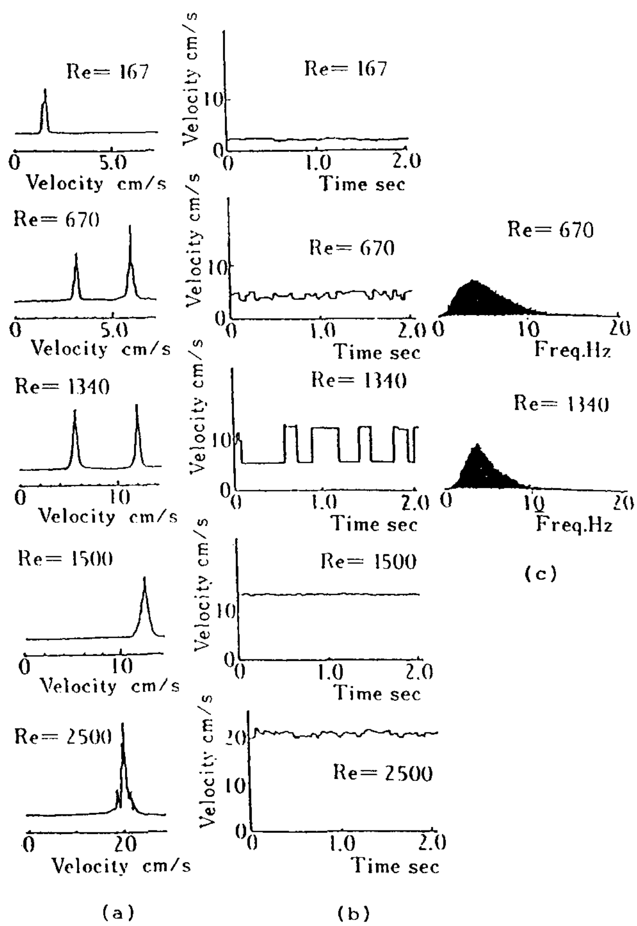


Figure 9. Velocity fluctuations obtained experimentally by Ohba *et al.*[18] in long-radius bend: (a) probability density function; (b) velocity as a function of time; (c) autospectral density function

of 520. At least some of the differences cannot be readily explained within the framework of the present numerical experiments.

4. CONCLUSIONS

The nature of the bifurcation solution in laminar flow in curved ducts of square cross-section has been investigated numerically using the HARWELL-FLOW3D computer program. The solution is found to be sensitive to the differencing schemes and grid refinement. In all three curved duct geometries investigated here the transition from two- to four-vortex structure occurred at a Dean number of about 110. At higher Dean (Reynolds) numbers the four-cell structure reverted to a two-cell structure through a route involving asymmetric flow patterns. This route appears to be very sensitive to the grid, duct geometry and numerical methods of solution. The results from this study are in agreement, at least in parts, with recent experimental,[17-19] theoretical[13] and numerical[12] results. It is suggested that the sensitivity of the solution to numerical accuracy, especially to the differencing scheme, and to the imposition of a symmetry boundary condition at the midplane in some of the earlier studies may explain some of the differences among the various numerical results.

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